IRREGULAR MOVEMENT OF GROUND WATERS WITH EVAPORATION

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The problem of variation in the level of ground waters between two vertical channels is examined, taking into account the evaporation, which is nonlinearly dependent on the level of the ground waters. The problem is reduced to integration of a diffusion equation with a right-hand side which is nonlinearly dependent on the unknown function and is solved by a method of successive approximations. When a certain inequality is achieved, which depends on the magnitudes entering the conditions of the problem, the method of successive approximations converges.

We will examine a problem of variation in the level of ground waters between two vertical channels, which are at a distance L from each other, taking evaporation into account. We will suppose that the intensity of evaporation of $\sigma f(h)$ is a given nonlinear function which is continuous with its derivative. We will also consider that at the initial moment of time the water levels in the channels suddenly become equal to H_1 and H_2.

This problem is reduced to integration of the diffusion equation with the nonlinear right-hand part

$$\frac{\partial h}{\partial t} = a^2 \frac{\partial^2 h}{\partial x^2} + f(h) \qquad \left(a^2 = \frac{\varkappa H_*}{\sigma}\right) \tag{1}$$

boundary conditions

$$h(0, t) = H_1, \quad h(L, t) = H_2 \tag{2}$$

and the initial condition

$$h(x, 0) = H_0, \qquad (0 < x < L) \tag{3}$$

Here h is the level of the ground waters, σ is the inadequacy of saturation or water delivery, H_* is the average depth of the ground flow, and $\varkappa$ is the filtration coefficient.

For a particular form of the function $f(h)$ the problem described by Eqs. (1)-(3) is examined in [1].

Hence it is assumed that the function $f(h)$ is constant or depends linearly on the difference $h-h_0$ if $h>h_0$, where h_0 is a certain critical level of ground waters, and if the ground waters occur sufficiently deeply $(h<h_0)$, then $f(h)\equiv 0$ (this can be neglected with evaporation).

We will assume below that the level of the ground waters exceeds the critical level h_0 $(h > h_0)$.

Let $h_0(x)$ be the steady-state solution of Eq. (1) with boundary conditions (2), i.e., the solution of the problem

$$a^2 \frac{d^2 h_0}{dx^2} + f(h_0) = 0, \qquad h_0(0) = H_1, \qquad h_0(L) = H_2 \tag{4}$$

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It is easily seen that the function h_0 (x) is determined by the relation

$$x = \pm \int_{H_1}^{h_0} \frac{du}{\sqrt{c - J(u)}} \qquad \left(J(u) = \frac{2}{a^2} \int_{H_1}^{u} f(v)\,dv\right) \tag{5}$$

where c is the root of the equation

$$L = \pm \int_{H_1}^{H_2} \frac{du}{\sqrt{c - J(u)}} \tag{6}$$

Introducing a new function z for h(x, t):

$$h(x, t) = h_0(x) + z(x, t) \tag{7}$$

on considering (4), we reduce the problem (1) to (3) to the following problem in the same region $0 < x < L$:

$$\frac{\partial z}{\partial t} = a^2 \frac{\partial^2 z}{\partial x^2} + f[h_0(x) + z] - f[h_0(x)] \tag{8}$$

$$z(0, t) = z(L, t) = 0, \qquad z(x, 0) = H_0 - h_0(x) \tag{9}$$

The solution of this problem will be sought by a method of successive approximations.

The following inequalities are established:

$$|f(u)| \leqslant M, \qquad |f'(u)| \leqslant M \tag{10}$$

We demonstrate the convergence of the method of successive approximations in the case of

$$^2/_3 ML^2 / a^2 = q < 1 \tag{11}$$

Designating for brevity

$$\Phi(z, x) = f[h_0(x) + z] - f[h_0(x)], \qquad \Omega(x) = H_0 - h_0(x) \tag{12}$$

we examine the succession of the approximations $z_n(x, t)$ determined on considering (8) and (9) by the relationships

$$\frac{\partial z_{n+1}}{\partial t} = a^2 \frac{\partial^2 z_{n+1}}{\partial x^2} + \Phi(z_n, x), \quad \begin{array}{l} z_{n+1}(0, t) = z_{n+1}(L, t) = 0 \\ z_{n+1}(x, 0) = \Omega(x) \end{array} \tag{13}$$

We will find the solution of the linear problem (13) in the form of the summation

$$z_{n+1} = u + v_{n+1} \tag{14}$$

Here u is the solution of the diffusion equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad \begin{array}{l} u(0, t) = u(L, t) = 0 \\ u(x, 0) = \Omega(x) \end{array} \tag{15}$$

and v_{n+1} is the solution of the inhomogeneous equation with the homogeneous boundary and initial conditions

$$\frac{\partial v_{n+1}}{\partial t} = a^2 \frac{\partial^2 v_{n+1}}{\partial x^2} + \Phi(z_n, x) \quad \begin{array}{l} v_{n+1}(0, t) = v_{n+1}(L, t) = 0 \\ v_{n+1}(x, 0) = 0 \end{array} \tag{16}$$

We will look for the solution u of Eq. (15) with boundary conditions in the form

$$u = \sum_{k=1}^{\infty} C_k \exp \frac{-\pi^2 a^2 k^2 t}{L^2} \sin \frac{\pi k x}{L} \tag{17}$$

Here the coefficients C_k are determined from the initial condition (15). We obtain

$$C_k = \frac{2}{L} \int_0^L \Omega(x) \sin \frac{\pi k x}{L}\,dx \tag{18}$$

The function v_{n+1} will be sought in the form of the series

$$v_{n+1} = \sum_{k=1}^{\infty} D_k^{(n)}(t) \exp \frac{-\pi^2 a^2 k^2 t}{L^2} \sin \frac{\pi k x}{L} \tag{19}$$

It is seen from this formula that the boundary conditions (16) are satisfied. It follows from the initial conditions (16) that the unknown functions $D_k^{(n)}(t)$ must fulfill the conditions

$$D_k^{(n)}(0) = 0 \quad (k = 1, 2, \ldots, n = 0, 1, 2, \ldots) \tag{20}$$

We will expand the function $\Phi(z_n, x)$ into the Fourier sine series

$$\sum_{k=1}^{\infty} f_k^{(n)}(t) \sin \frac{\pi k x}{L}, \quad f_k^{(n)}(t) = \frac{2}{L} \int_0^L \Phi[z_n(y, t), y] \sin \frac{\pi k y}{L} dy \tag{21}$$

Substituting into Eq. (16) the series (19) and (21), we obtain

$$\frac{dD_k^{(n)}}{dt} = f_k^{(n)}(t) \exp \frac{\pi^2 a^2 k^2 t}{L^2} \quad (k = 1, 2, \ldots, n = 0, 1, 2, \ldots) \tag{22}$$

Integrating Eq. (22) and taking into account the condition (20), we find

$$D_k^{(n)}(t) = \int_0^t f_k^{(n)}(\tau) \exp \frac{\pi^2 a^2 k^2 \tau}{L^2} d\tau \tag{23}$$

Now, substituting Eq. (23) in Eq. (19) and taking the expression (21) into account for $f_k^{(n)}(t)$, we find the final equation for determining the function $v_{(n+1)}(x, t)$:

$$v_{n+1}(x, t) = \frac{2}{L} \sum_{k=1}^{\infty} \int_0^t \int_0^L \Phi[z_n(y, \tau), y] \exp \frac{-\pi^2 a^2 k^2 (t - \tau)}{L^2} \sin \frac{\pi k x}{L} \sin \frac{\pi k y}{L} dy\, d\tau \tag{24}$$

It is easy to see that the series (24) converges uniformly (its general term does not exceed the magnitude $4ML^2/\pi^2 a^2 k^2$).

Assuming that the nth-approximation $z_n(x, t)$ is known, $(n+1)$th approximation will be determined by Eqs. (14), (17), (18), and (24) using the designations (12).

We will demonstrate that all the approximations $v_n(x, t)$ are limited by the same constant.

On considering (10), the inequality follows from (24) (the range of the continuous functions C is examined):

$$\|v_{n+1}(x, t)\| \leqslant q, \quad q = \frac{4MSL^2}{\pi^2 a^2} \quad \left(S = \sum_{k=1}^{\infty} \frac{1}{k^2}\right) \tag{25}$$

As is already known [2], the sum $S = (1/6)\pi^2$. Consequently (11) results from Eq. (25).

We will demonstrate the convergence of the method of successive approximations in the case $q < 1$.

For the difference of two consequent approximations, we obtain from (24) the following expression:

$$v_{n+1}(x, t) - v_n(x, t) = \frac{2}{L} \sum_{k=1}^{\infty} \int_0^t \int_0^L \{\Phi[z_n(y, \tau), y] - \Phi[z_{n-1}(y, \tau), y]\} \times \exp \frac{-\pi^2 a^2 k^2 (t - \tau)}{L^2} \sin \frac{\pi k x}{L} \sin \frac{\pi k y}{L} dy\, d\tau \tag{26}$$

According to the mean value theorem,

$$\Phi[z_n(y, \tau), y] - \Phi[z_{n-1}(y, \tau), y] = \Phi_\theta'[\theta, y](z_n(y, \tau) - z_{n-1}(y, \tau)) \tag{27}$$

where θ is situated between z_{n-1} and z_n. From (27), (26), and (10) and also using (14), we consequently find

$$\|v_{n+1} - v_n\| \leqslant q \|z_n - z_{n-1}\|, \quad \|v_{n+1} - v_n\| \leqslant q \|v_n - v_{n-1}\| \tag{28}$$

From (28) it is seen that in the case $q < 1$, the series converges uniformly. The evaluation

$$\|v_{n+1} - v_n\| \leqslant q^n \|v_1 - v_0\|, \quad \|z_{n+1} - z_n\| \leqslant q^n \|z_1 - z_0\| \tag{29}$$

follows from the inequalities (28).

The level of the ground waters $h(x, t)$ is determined on considering (7), (14), and (24) by the relationship

$$h(x, t) = h_0(x) + u(x, t) + \lim_{n \to \infty} v_n(x, t) \tag{30}$$

where u (x, t) is given by the relationships (17) and (18) and v_{n+1} (x, t) is given by Eq. (24).

It is easily seen that if the intensity of evaporation is a constant magnitude $f(h) = -\alpha$, then the depth of the ground flow h (x, t) is represented by the following expression:

$$\begin{gathered} h(x, t) = h_0(x) + \sum_{k=1}^{\infty} a_k \exp \frac{-\pi^2 a^2 k^2 t}{L^2} \sin \frac{\pi k x}{L} \\ h_0(x) = \frac{\alpha}{2a^2} x^2 + \left(\frac{H_2 - H_1}{L} - \frac{\alpha L}{2a^2} \right) x + H_1 \\ a_k = \frac{2}{\pi} \left\{ \frac{H_0 - H_1 + (-1)^k (H_2 - H_0)}{k} + \frac{\alpha L^2}{a^2} \frac{[1 - (-1)^k]}{k^3} \right\} \end{gathered} \tag{31}$$

Now let $f(h) = -\beta h$ (the intensity of evaporation is a linear function of the depth of the ground flow, calculated from the bottom of the reservoir).

Using the Eqs. (17), (24), (5), (12), and (18) and approaching the limit in Eq. (30), we find the relationship between the level of ground waters and x and t:

$$\begin{gathered} h(x, t) = h_0(x) + \sum_{k=1}^{\infty} b_k \exp \left[-\left(\beta + \frac{\pi^2 a^2 k^2}{L^2} \right) t \right] \sin \frac{\pi k x}{L} \\ h_0(x) = \frac{H_1 \operatorname{sh} \lambda (L - x) + H_2 \operatorname{sh} \lambda x}{\operatorname{sh} \lambda L} \qquad \left(\lambda = \frac{\sqrt{\beta}}{a} \right) \\ b_k = \frac{2}{\pi k} \left\{ H_0 - H_1 + (-1)^k (H_2 - H_0) + \frac{\beta [H_1 - (-1)^k H_2]}{\beta + \pi^2 a^2 k^2 / L^2} \right\} \end{gathered} \tag{32}$$

The direct integration of Eq. (1), in which $f(h) = -\beta h$, with boundary conditions (2) and the initial condition (3) leads to the same result (32), which is known in the theory of filtration.

LITERATURE CITED

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